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Problems, problems, problems

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> At a exhibition of electronic computing machines in the Netherlands (many years ago) Queen Juliana remarked that not only could she not understand these machines but she could not understand the people who could understand them.

As chairman of the two "Problems" sessions at the First Canadian Conference on Computational Geometry (FCCCG), I undertook the task of preparing a record of all the presentations as well as some additional problems submitted in writing but not presented at the conference. These are all given here in alphabetical order by author.

> Here's to Dr. Avis, D.,
> He planned the FCCCG
> You'll know him by his unshorn locks
> And by his frequent lack of socks.
>
> *W. Moser*

D. Avis (McGill University) and **H. Imai** (Kyushu University)

Let S be a set of n points in the plane, no 4 co-cyclic, and let

$$\{\theta_i \mid 0<\theta_1<\theta_2<\cdots<\theta_{k-1}<2\pi\}$$

be a set of $k-1$ angles. A point x in the plane is a *valid placement* if there exists

$$\{s_0, s_1, \ldots, s_{k-1}\} \subseteq S$$

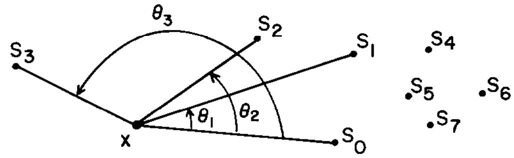
such that

$$\angle s_0 x s_i = \theta_i, \quad i=1,2,\ldots,k-1.$$

For fixed k, n, let $f_k(n)$ be the maximum number of valid placements for any set of n points and k angle measurements. It is known that:

$$f_4(n)=\Theta(n^3); \qquad f_k(n)=\Omega(n^2) \quad k \text{ fixed}; \qquad f_k(n)=O\left(\frac{n^3}{k}\right), \quad k\geq 5.$$

Obtain tighter bounds for $k\geq 5$.



A valid placement ($k=4$).

Peter Eades (University of Queensland) and **Roberto Tamassia** (Brown University)

A directed graph G is *upward planar* if it can be drawn so that

(a) there are no arc crossings, and
(b) all arcs are monotonically increasing in the y direction (i.e., they point upward).

From (a), the underlying undirected graph of G must be planar, and from (b) G must be acyclic. Unfortunately these two necessary conditions are not sufficient. We would like to know the complexity of the following problem.

Upward Planarity Test Problem.
Instance. An acyclic directed graph G with planar underlying undirected graph.
Question. Is G upward planar?

P. Erdős (Hungarian Academy of Sciences)

Problems on distances determined by a finite set of points

I stated many problems in this subject and even obtained a few results, some of which were greatly strengthened by interested colleagues. Here I do not try to give a survey of all the problems and results but try to state a few newer ones and the sharpest conjectures which I can think of. I will mostly restrict myself to the plane because this is perhaps the most interesting.

Let S denote a finite set of points—in the plane unless stated otherwise. Denote by $D(S)$ the number of distinct distances determined by the (pairs of) points of S. I conjectured more than 40 years ago [4] that

$$D(n)=\min_{|S|=n} D(S)>\frac{cn}{\sqrt{\log n}} \tag{1}$$

(where the minimum is taken over all possible sets of n distinct points). I offered (and offer) \$500 for a proof or disproof of (1). The lattice points show that (1), if true, is best possible apart from the value of the constant c. We say "S implements $D(n)$" if $|S|=n$ and $D(S)=D(n)$. It is not impossible that, for $n>n_0$, $D(n)$ is implemented by a subset of the triangular lattice. Perhaps this guess is completely wrongheaded. The regular pentagon shows that it is false for $n=5$, so it might be interesting to try to find other counterexamples. The currently best known lower bound for $D(n)$ is $D(n)>cn^{4/5}$ due to Chung, Szemerédi and Trotter [3].

Denote by $d(S,p)$ the number of distinct distances from $p\in S$ to the other points in S. I conjecture a much sharper result than (1), namely that

$$\max_{p\in S} d(S,p)>\frac{cn}{\sqrt{\log n}} \quad \text{if } |S|=n, \tag{2}$$

and in fact that, if $|S|=n$, then the number of points $p\in S$ for which $d(S,p)>cn/\sqrt{\log n}$ is $>c'n$. This conjecture is perhaps too optimistic. It would also be interesting to determine

$$F(n)=\min_{|S|=n}\sum_{p\in S} d(S,p).$$

Perhaps $F(n)=(c+o(1))n^2/\sqrt{\log n}$ and $F(n)$ is implemented by a subset of the triangular lattice.

It would be of interest to obtain a result of the following type. Let $h(n)\to\infty$ (e.g., $h(n)=n^{\alpha}$). What is the maximum number of points $p\in S$ for which $d(S,p)<h(n)$?

Assume that every subset of 4 points of S determines at least 5 different distances. Is it then true that $D(S)>cn^2$ when $|S|=n$? In fact I conjecture with some trepidation that

$$\exists\, T\subset S,\ |T|>c_1 n, \text{ and all distances determined by the points of } T \text{ are distinct} \tag{3}$$

(i.e., no distance is repeated). More generally consider the hypergraph formed by the quadruplets (of points in S) which determine exactly 5 different distances. Perhaps this hypergraph is 2-chromatic, or more modestly has bounded chromatic number. If our hypergraph is really 2-chromatic, then of course (3) is true with $c_1\geq\frac{1}{2}$.

I want to state one of my old \$500 conjectures [4]. Let $r(S)$ denote the maximum number of times the same distance occurs (among the distances determined by the points of S). I conjectured that

$$r(S)<n^{1+c/\log\log n} \quad \text{if } |S|=n. \tag{4}$$

The lattice points show that if (4) is true, then it is best possible.

I conjectured a few years ago that if $|S|=n$, $n>n_0$ and no 3 points of S are on a line and no 4 are on a circle, then it cannot happen that the most frequent distance occurs $n-1$ times and the ith (with respect to frequency of occurrence) distance occurs $n-i$ times, $1\le i\le n-1$. Palasti [8] proved that if this conjecture is true we must have $n_0\ge 9$. A stronger conjecture states:

> If $|S|=n$ and no 3 points are on a line and no 4 are on a circle, then $D(n)/n\to\infty$.

In fact I could not even prove $D(n)\ge n$ under these conditions and do not have a non-trivial bound for $D(n)/n$. Pach showed that $D(n)<n^{\log 3/\log 2}$. Recently Pach and Ruzsa have improved this bound to $D(n)=O(n\,e^{c\sqrt{\log n}})$, but no superlinear lower bound is known.

Now we consider some problems when the set S is convex (i.e., when the points of S are the vertices of a convex polygon), where there is more hope to get sharp results. I conjectured and Altman [1,2] showed that

$$D(S)\ge[\tfrac{1}{2}n]\quad \text{if } |S|=n \text{ and } S \text{ is convex.} \tag{5}$$

Szemerédi conjectured that

$$D(S)\ge[\tfrac{1}{2}n]\quad \text{if } |S|=n \text{ and no 3 points of } S \text{ are collinear}$$

but he only proved that $D(S)\ge\frac{1}{3}n$ in this case. I also conjectured that for at least one $p\in S$

$$d(S,p)\ge[\tfrac{1}{2}n]. \tag{6}$$

This is still open but will perhaps not be too hard to prove. I hope that the number of points $p\in S$ for which (6) holds is $>cn$, and perhaps a sharp inequality for $\sum_{p\in S}d(S,p)$ can be obtained.

I conjectured that every convex polygon has a vertex which has no 3 other vertices equidistant from it. This conjecture was disproved by Danzer, who constructed a convex 9-gon in which every vertex has 3 other vertices at the same distance from it. I than conjectured that there is some r (hopefully $r=4$) so that every convex polygon has a vertex from which no r vertices are equidistant. I offer \$250 for a proof or disproof of this fascinating conjecture.

Let $h(n)$ denote the maximum number of times the same distance can occur in a convex n-gon. More than 30 years ago Moser and I [6] conjectured that $h(n)\le cn$ (c an absolute constant). We observed that $c\ge\frac{5}{3}$. Edelsbrunner and Hajnal proved that $h(n)\ge 2n-7$. By the way Füredi recently proved that $h(n)<cn\log n$. If my conjecture (in the previous paragraph) holds we would have $h(n)<rn$.

Let us now assume that we have n points on the 3-dimensional unit sphere. Erdős, Hickerson and Pach [5] proved that no distance can occur more than $cn^{4/3}$ times, but the distance $\sqrt{2}$ can actually occur $n^{4/3}$ times. Every distance <2 can occur $n\log^*n$ times, where $\log^*n$ denotes the iterated logarithm function. We could not get any better result for any distance $\ne\sqrt{2}$. Perhaps if we only assume that the n

points are the vertices of a convex polyhedron stronger results will hold.

Perhaps more interesting is the following question. If S is a set of n points on the unit sphere, is it true that $D(S) > cn$? (Is $D(S) \geq n$ if $|S| = 2n+2$? Note that equality is attained when S contains the 2 poles and the vertices of a regular $2n$-gon on the equator.) We could get nowhere with this interesting problem.

We conclude with an old question of Donald Newman and myself. Let there be given in the unit square n^2+1 nonoverlapping squares of sides $x_1, x_2, \dots, x_{n^2+1}$. Is it then true that

$$\sum_{i=1}^{n^2+1} x_i \leq n? \tag{7}$$

This holds trivially for n^2 squares, and it is easy to see that it does not hold for n^2+2 squares. (Also, (7) is not hard to show for $n=1$.)

Many more related new questions are asked in my forthcoming paper with Pach [7].

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[7] P. Erdős and J. Pach, Variations on the theme of repeated distances, Combinatorica, to appear.

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A problem both deep and profound,
Is whether a circle is round.
In a paper by Erdős
Written in Kurdish
A counter-example is found.

Jacob E. Goodman (City College, CUNY)

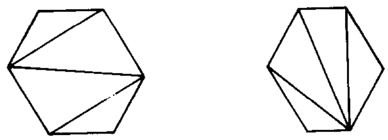
In a convex d-polytype Δ in $\mathbb{R}^d$, consider (for $1 \leq k \leq d-1$) the ratio $\sigma_k(\Delta)$ between the k-volume of a maximal-volume k-face of Δ and that of a minimal-volume k-face. Define $\tilde{\sigma}_k(\Delta)$ as the minimum of $\sigma_k(\Delta')$ over all polytopes Δ' combinatorially equivalent to Δ. What bounds can be given for $\tilde{\sigma}_k(\Delta)$ if Δ has n vertices?

Note: If one replaces "polytope" by "configuration of points in general position", "k-face" by "simplex", and "combinatorial equivalence" by "same order type", it is possible to prove both doubly-exponential upper and lower bounds in the analogous quantity. In particular, this implies a doubly-exponential upper bounds for $\tilde{\sigma}_k(\Delta)$, but this may be far from the truth.

> A mathematician gave a talk on genetics to a group of mathematicians and geneticists. He began by describing his proposed method of procedure. "First," he said "I will have to explain the genetic aspects of the problem to the mathematicians. After that I will go on to explain the mathematical aspects of the problem to the mathematicians."

L. Guibas (Digital Equipment Corp., System Research Center)

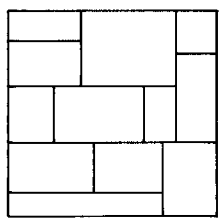
(1) Let a convex polygon C be given in the xy-plane, along with two triangulations of it: T_1 and T_2.



Assume further that T_1 and T_2 have no common edges.

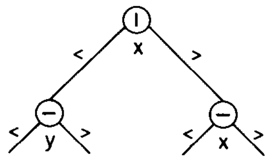
Conjecture: It is always possible to perturb the vertices of C vertically out of the xy-plane (i.e., by displacements parallel to the z-axis) so that the polygon C becomes a spatial polygon C' such that the convex hull of C' is a convex polyhedron consisting of two triangulated cups glued along C', *and* the triangulation of the upper cup (i.e., those faces oriented toward $+z$) is that specified as T_1, and the triangulation of the lower cup is that specified as T_2.

(2) Let S be a subdivision of the unit square into n rectangles oriented parallel with the axes.



Consider algorithms that perform point-location in S that are modelled by finite trees based on x-tests or y-tests.

(An x-test compares the x-coordinate of the query point against one of the x-coordinates of the vertical edges present in S; and similarly for y-tests.)



Tree T

Conjecture: There does not exist a point-location tree T with $O(n)$ nodes and maximum depth $O(\log n)$.

Note: At each leaf of T we must know the region of S containing the query point.

H. Maehara (Ryukyu University)

Is there a finite simple graph G which cannot be represented by an *integral distance graph* in $\mathbb{R}^2$? (An integral distance graph in $\mathbb{R}^2$ is a graph with vertices in $\mathbb{R}^2$ in which two vertices x, y are adjacent if and only if $\|x-y\| \in \mathbb{Z}_+$.)

Note: $K_{m,n}$ can be represented by an integral distance graph in $\mathbb{R}^2$.

John McKay (Concordia University)

Let P be a finite set of points in the plane. For any point $S \notin P$ let

$$f(P,S) = \min_{X,Y \in P} \angle XSY \quad \text{and} \quad F(P) = \max_{S \notin P} f(P,S).$$

What can be said about the points S for which $f(P,S) = F(P)$? When is there a unique S? Generalize to n-space (and "solid" angles).

A. Meir (University of Alberta)

(1) Given $p>0$, $q>0$. Suppose $\{a_n\}$ satisfies $\|a_n\|_p = \infty$. Show that $\exists \{b_n\} \in l_q$ such that if $\{c_n\}$ is *any* sequence in l_q, then $\|a_n \cdot (b_n/c_n)\|_p = \infty$. (Solved.)

(2) (A fun excercise) Let $n_1, n_2, \ldots, n_k$ be integers with $\gcd\{n_1, \ldots, n_k\} = 1$. Show that $\exists B>0$ such that

$$\left| 1 + \sum_{j=1}^{k} e^{in_j\theta} \right| < 1 + k - B\theta^2$$

for every $\theta \in [-\pi, \pi]$.

(3) Let S be any set of n points in the unit square. It is known (D.J. Newman: Problem Seminar) that there exists a Hamiltonian circuit $P_1, \ldots, P_n, P_1$ such that if $d_j = \mathrm{dist}(P_j, P_{j+1}), j = 1, 2, \ldots, n$, then

$$\sum_{j=1}^{n} d_j^2 \leq 4. \tag{1}$$

Newman's proof is *not* constructive; find an algorithm which gives

$$\sum d_j^2 < C \text{ (constant)},$$

for any S.

(The "nearest neighbour algorithm" gives $\sum d_j^2 \sim C \log n$.)

Open problem: Is the generalization of (1) true in $\mathbb{R}^k$, i.e., $\sum_{j=1}^{k} \leq 2^k$?

V. Milenkovic (Harvard University)

We are given n lines $L_1, L_2, \ldots, L_n$ with positive slope, ordered by increasing slope.

Let $\langle X_{ij}, Y_{ij} \rangle$ be the intersection of L_i and L_j, $1 \leq i < j \leq n$.

Let p_{ij}, q_{ij} be real numbers uniformly chosen from the interval [0.75,1.25].

For $1 \leq i < j \leq n$ let

$$A_{ij} = X_{ij} - \frac{p_{ij}}{\theta_{ij}}, \qquad B_{ij} = X_{ij} + \frac{q_{ij}}{\theta_{ij}},$$

where θ_{ij} is the angle between lines L_i and L_j.

It can be shown that

(1) $A_{ij} \leq X_{ij} \leq B_{ij}$ for $1 \leq i < j \leq n$,
(2) $X_{ij} \leq X_{ik} \leq X_{jk}$ or $X_{jk} \leq X_{ik} \leq X_{ij}$ for $1 \leq i < j < k \leq n$.

The problem is this. Given only the order of the set

$$\{A_{ij}\}_{1 \leq i < j \leq n} \cup \{B_{ij}\}_{1 \leq i < j \leq n}$$

determine a set of values X_{ij}, $1 \leq i < j \leq n$, that satisfy (1) and (2). We are not allowed to know the L_i or the X_{ij}.

W. Moser (McGill University)

During his all too brief life, Leo Moser (1921–1970) created and collected many problems in number theory, graph theory, combinatorics and combinatorial geometry. In 1966 he circulated a mimeographed collection of 50 problems: *Poorly formulated problems of combinatorial geometry* [44]. Eight years later Richard Guy

(1975) appended a selection of 30 of these problems (a few of them edited) to the problems section of the published proceedings of a conference. Leo's 50 problems stimulated much research, and many papers on these problems have appeared during the intervening 25 years. In the hope that participants and readers will help me to bring the record up to date, I present here the 50 problems, in the following format. For each problem (numbered LM 1 (1966), LM 2 (1966), ..., LM 50 (1966)) I will state the problem *precisely* as given by Leo (see [44]), signifying its end with a □. Each *original* version will be followed immediately by a restatement (if necessary, and signifying the end by □□) and information I have available about the problem at this time.

I will be happy to receive new information from readers, particularly the location of papers which seem to be relevant. Perhaps 10 years from now (in the year 2000) I shall again prepare an update on the state of Leo Moser's 1966 collection.

There was a mathematician named Moser,
Well known as a problem proposer.
He gave some that were silly
To his brother named Willy,
Did that stump him? The answer is No Sir!

LM 1 (1966). The set of squares of sides $d_1 \geq d_2 \geq \cdots$, $\sum d_i^2 = 1$, can be placed in a single square of side $d_1 + \sqrt{1 - d_1^2} \leq \sqrt{2}$. Extend this "best possible" result to n dimensions. □

The assertion in the first sentence was proved by Meir and Moser [41]. See comments in LM 8 (1966).

LM 2 (1966). Can one obtain better estimates for the smallest square which can accomodate squares of sides $d_1 \geq d_2 \geq \cdots$, i.e., estimates involving more or perhaps all of the d's. □

See comments in LM 8 (1966).

LM 3 (1966). Can every set of rectangles of total area 1 and maximal side 1 be accomodated in a square of area 2? □

See comments in LM 8 (1966).

LM 4 (1966). Can the rectangles of sides $(1/n, 1/(n+1))$, $n = 1, 2, 3, \ldots$ be accomodated in a unit square? □

See comments in LM 8 (1966).

LM 5 (1966). Can any set of rectangles of largest edge 1 and total area 3 be used to cover a unit square? (No rotations, please.) □

See comments in LM 8 (1966).

LM 6 (1966). The squares of sides $\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots$ can be accomodated in a square of side $\frac{5}{6}$ and this is "best possible". Can they be accomodated in a rectangle of area $\frac{1}{6}\pi^2 - 1$? □

See comments in LM 8 (1966).

LM 7 (1966). What is the smallest number A such that every set of squares of total area 1 can be accomodated in some rectangle of area A? □

See comments in LM 8 (1966).

LM 8 (1966). Can the squares of sides $1, 2, \ldots, 24$ be accomodated in a square of side 70 ($1^2 + 2^2 + \cdots + 24^2 = 70^2$)? (Almost certainly not.) □

The set of squares of sides $d_1 \geq d_2 \geq d_3 \geq \cdots$, $\sum d_i^2 = 1$, can be placed without overlap in a square of side

$$d_1 + \sqrt{1 - d_1^2} \leq \sqrt{2}.$$

This result is contained in Moon and Moser [42]. Can one obtain better estimates involving more of the d_i's? Extend the results to n dimensions. Meir and Moser [41] obtained the following generalization. Suppose $x_1 \geq x_2 \geq \cdots$ are the sides of cubes in k-dimensional space and $a_1, a_2, \ldots, a_k$ are the edges of a rectangular parallelepiped; then it is possible to pack the cubes into the parallelepiped if

$$a_j \geq x_1, \quad j = 1, 2, \ldots, k,$$

and

$$x_1^k + (a_1 - x_1)(a_2 - x_2) \cdots (a_k - x_k) \geq V,$$

where V denotes the volume of the cubes. Other results in Meir and Moser [41] deal with packing rectangles into rectangles, and:

(i) all squares of sides $\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots$ can be packed into a square of side $\frac{5}{6}$ (a best possible result);

(ii) all rectangles of size $1/k \times 1/(k+1)$, $k = 1, 2, \ldots$, can be packed into a square of side $\frac{31}{30}$.

Open questions remain.

(a) What is the smallest number S such that any set of squares of total area 1 may be packed in a rectangle of sides 1 and S? Note: $\sqrt{3} \leq S \leq 2$ [42].

(b) What is the smallest number T such that any set of squares of total area 1 may be packed in some rectangle of area T? Note: $1.2 < T \leq 2$ [42].

(c) What is the area R of the smallest rectangle in which can be packed the set of rectangles of side lengths $1/n$, $1/(n+1)$, $n = 1, 2, 3, \ldots$? (Of course $R \leq \frac{31}{30}$.) Is $R > 1$?

A survey paper by Göbel [30] is relevant. In Gardner [28] the following problem was proposed: Can the squares of side $1, 2, 3, \ldots, 24$ be packed in a square of side 70? (Note that $1^2 + 2^2 + 3^2 + \cdots + 24^2 = 70^2$.) An exhaustive computer search by E.M. Reingold and J. Bitner showed that the answer is NO [29].

Kosinski [39] proved that a collection of convex bodies with uniformly bounded diameter can be packed in a bounded region of $\mathbb{R}^n$.

(d) Coxeter [17] asks: Find the radius of the smallest circle inside which discs of radius $1/n$ $(n=1,2,3,\ldots)$ can all be packed.

(e) n nonoverlapping spheres (in E^d) of radii $r_1,\ldots,r_n$ are packed in a unit hypercube. Find bounds on

$$r_1+r_2+\cdots+r_n.$$

In $E^2, r_1+r_2+\cdots+r_n<\sqrt{n}/12^{1/4}$ and the constant is the best possible.

For a related covering problem see Newman [46].

LM 9 (1966). What is the region of smallest area which will accomodate every arc of length 1? □

See comments in LM 50 (1966).

LM 10 (1966). Will the semi-ellipse $12x^2+16y^2=3$ accomodate every arc of length 1? □

See comments in LM 50 (1966).

LM 11 (1966). What is the largest number $f=f(a,b,c)$ such that every closed curve of length f can be accomodated in the triangle (if it exists) of sides a,b,c? How is it for arcs? □

See comments in LM 50 (1966).

LM 12 (1966). Find or estimate the smallest number $f(n)$ of points in the plane such that one can always pick $f(n)$ points from any n noncollinear ones from which all the remaining points are visible. Conjecture: $f(n)=O(\log n)$. □

LM 13 (1966). Estimate the size of the largest circle which can be placed in the square lattice so as not to be nearer than ε to any lattice point. □

Erdős comments: c or ε has perhaps been proved by Beck.

LM 14 (1966). Given n points on a sphere. It is conjectured that the same distance can occur at most $3n-6$ times. □

The conjecture is false. Erdős, Hickerson and Pach [22] proved ($\log^* n$ denotes the smallest integer r such that, starting with n, one has to iterate the logarithm function r times to get a value smaller than or equal to 1): There exist $c_1, c_2>0$ such that

(i) for every natural number n and for every $0<\alpha<2$ one can find n points in S^2 with the property that each is at distance α from at least $c_1 \log^* n$ others;

(ii) for every natural number n one can find n points in S^2 with the property that each is at distance $\sqrt{2}$ from at least $c_2 n^{1/3}$ others.

LM 15 (1966). What is a region of largest area which can be moved through a hallway of width 1? □

LM 16 (1966). Let P_i, $i=1,2,3,\ldots,n$ be n points on a unit sphere. Then $\sum_{i\geq j}\overline{P_iP_j}^2\leq n^2$ if and only if the centroid of the points is at the center of the sphere. What is the corresponding inequality for $\sum\overline{P_iP_j}$? □

Let $P=\{p_1,p_2,\ldots,p_n\}$ be a finite set of points on the unit sphere in k-dimensional Euclidean space,

$$f(P,k)=\sum_{1\leq i<j\leq n}\overline{p_ip_j}$$

and

$$f(n,k)=\max_{|P|=n} f(P,k).$$

The problem is to determine (bounds on) $f(n,k)$ and extreme configurations.

In 2 dimensions the extreme configuration is the regular n-gon [26] with sum equal to $n\cot(\pi/(2n))$. It is also known that if $n=k+1$, then the regular simplex is optimal (see Hille [36]), e.g., $f(4,3)=9.79796$.

For $k\geq 3$, $n>k+1$ the exact value of $f(n,k)$ is unknown. The reason for this is that if n is large enough (compared to k), then there are no "regular" configurations on the sphere, so the central point-system(s) is (are) expectedly quite complicated and "irregular". For the study of this case Stolarsky [57, 58] suggested the following fruitful and ingenious approach.

Let S^{k-1} denote the unit sphere in E^k whose surface area is denoted by $\sigma(S^{k-1})$. Let $p_0\in S^{k-1}$ be a fixed point and set

$$A(n,k)=\frac{n^2}{2\sigma(s^{k-1})}\int_{S^{k-1}}\overline{p_0p}\,\mathrm{d}\sigma(p),$$

where $\mathrm{d}\sigma(\cdot)$ is the surface element. Observe that

$$A(n,k)=\frac{n^2}{2\sigma(S^{k-1})}\int_{S^{k-1}}\int_{S^{k-1}}\overline{pq}\,\mathrm{d}\sigma(p)\,\mathrm{d}\sigma(q),$$

so $A(n,k)$ can be regarded as the "continuous relaxation" of the sum $f(P,k)=\sum_{1\leq i<j\leq n}\overline{p_ip_j}$, where the points p_i have been dispersed and uniformly spread out on the sphere.

Put

$$\sigma(x)=\int_{\langle p_0,p\rangle\leq x}\mathrm{d}\sigma(p).$$

Let $SO(k)$ denote the (special orthogonal) group of rotations in E^k, and let $\mathrm{d}\tau$ be the Haar measure on $SO(k)$. Stolarsky discovered the following nice identity:

$$\int_{-1}^{+1}\int_{SO(k)}\left(f(P,\tau,x)-\frac{n\sigma(x)}{\sigma(S^{k-1})}\right)^2\mathrm{d}\tau\,\mathrm{d}x=A(n,k)-f(P,k),$$

where $f(P,\tau,x)$ denotes the number of those elements $p_i\in P$ for which $\langle p_i,\tau(p_0)\rangle\leq x$. ($\tau(p_0)$ is the image of p_0 under $\tau\in SO(k)$.) This means that in order to maximize $f(P,k)$, $|P|=n$, we have to minimize the above integral. Using this idea Stolarsky proved that

$$\min_{|P|=n} (A(n,k)-f(P,k)) \le c_k n^{1-1/(k-1)},$$

and Beck [5] showed that this bound is best possible apart from the exact value of the constant c_k. That is

$$f(P,k) \le A(n,k) - c'_k n^{1-1/(k-1)}.$$

(See also Harman [35].)

The values of c_k and c'_k are very far from each other if k is large.

Berman and Hanes [8] used a computer search to obtain configurations (for $k=3$ and small n) yielding (first row in table below).

Table 1

	n					
	5	6	7	8	9	10
$f(n,3) \ge$	15.6814	22.9706	31.5309	41.4731	52.7436	65.3497
$f(n,3) \le$	16.1667	23.5	32.1667	42.1667	53.5	66.1667

(The upper bounds in the last row are from Alexander [1].) They note that with the exception of $n=7$, the configurations (yielding the lower bounds) "which arose in our computer search closely agree with best known solutions for minimizing $\sum 1/(p_i - p_j)$. Consult Cohen [16] or Goldberg [31] for details. Also, Bjorck [11] contains a detailed discussion of these and related problems."

This problem and similar ones are of interest to chemists and physicists, see e.g., Cohen [16], Alexander and Stolarsky [2], Gardner and Radin [27], Hammrick and Radin [34].

Given P as before, and p any point (on the unit sphere) let

$$g(P,k,p,\lambda) = \sum_{i=1}^{n} \overline{pp_i}^{\lambda}, \quad 0<\lambda<2.$$

Stolarsky [58] asked: For what p is $g(P,k,p,\lambda)$ maximal (minimal)? He solved this for some particular choices of P.

The similar problem (maximizing the sum of distances) with Euclidean distance replaced by spherical distance (the shortest arc between points) is relatively easy: the sum of all the $k(2k-1)$ mutual distances of $2k$ points on the unit sphere is at most $k^2\pi$, and this maximum is achieved if k of the points coincide at the North pole and k coincide at the South pole. The short elementary proof of this by Sperling [55] extends to $2k$ points on the sphere in higher dimensions.

Also solved is the problem of minimizing the sum of distances when the convex hull of the n points (on the unit sphere) contains the origin. Here Chakerian and Klamkin [14] conjectured and Wolfe [63] proved that

$$\sum_{1 \le i<j \le n} \overline{p_i p_j} \ge 2n-2$$

with equality when $p_2 = p_3 = \cdots = p_n = -1$. This inequality was also proved by Chakerian and Ghandehari [13] in the more general setting of a real normed linear space.

LM 17 (1966). If P_i, $i = 1, 2, \ldots, n$ are n points with mutual distances $\overline{P_iP_j} \geq 2$ $(i \neq j)$, then (Blichfeldt) for any point O, $\sum \overline{OP_i}^2 \geq 2(n-1)$ with equality possible for $n = 2, 3, 4$. What is the corresponding sharp inequality for somewhat larger values of n? □

> L. Moser once pointed out to a class that given 23 people chosen at random the probability was greater than $\frac{1}{2}$ that two of these had the same birthday. This particular class had only 16 students. Nevertheless Leo offered even money that two of these students had the same birthday! (Leo's twin brothers were in the class!!)

LM 18 (1966). We can prove the following sharp inequality:

$$\max_{x_1^2 + x_2^2 + x_3^2 = 1} \quad \min_{\varepsilon = -1, 0, 1 \text{ (not all 0)}} |\varepsilon_1 x_1 + \varepsilon_2 x_2 + \varepsilon_3 x_3| \leq \frac{1}{\sqrt{21}}.$$

Be wise—generalize! □

LM 19 (1966). Prove that there is a function $f(n)$ which tends to infinity with n such that every region of area n can be placed on the square lattice (rotations allowed) so as to cover $n + f(n)$ lattice points. □

It is well known that any region of area n can be placed so as to cover n lattice points (see Steinhaus [56, p. 97]).

Wills [62] remarked that if there is a general function f for all convex regions $K \subseteq E^2$, then $f(n) = o(\sqrt{n})$ because of Pick [49] and Nosarzewska [47]:

$$V(K) - \frac{F(K)}{2} \leq G(K) \leq 1 + V(K) + \frac{F(K)}{2}, \quad K \subset E^d$$

($G(K) = \text{card}(K \cap Z^d)$; V = volume(area in E^2); F = surface area(perimeter in E^2)).

There is no analogue for translates of K. Indeed, if Q is a square, sides parallel to the coordinate axes, of area $q^2 - 1$ (q a positive integer), then every (any) translate of Q covers at most q^2 lattice points.

The particular case where K is a circular disc was solved by Skriganov [54] with $f(n) = x^{1/6-\varepsilon}$; he dealt with arbitrary lattices in E^2, not just with the usual square lattice Z^2.

Beck [6, 7] solved Moser's problem.

Beck's Theorem. There is a universal function $f(x)$, $f(x) \geq x^{1/9}$ for $x \geq c_0$ (c_0 is a "ineffective" absolute constant), such that any convex region K of area x can be placed on the plane so as to cover at least $x + f(x)$ (or at most $x - f(x)$) lattice points.

A modification of the proof yields the sharper lower bound

$$f(x) \geq x^{1/8-\varepsilon} \quad \text{for } x \geq c_0(\varepsilon).$$

Beck conjectures that the true order of magnitude of $f(x)$ is about $x^{1/4}$.

The problem in E^d, $d \geq 3$, remains open.

LM 20 (1966). If LM 19 is too easy, obtain good estimates for the largest such $f(n)$. □

LM 21 (1966). It is known that if all the faces of a convex polyhedron have central symmetry, then so has the polyhedron. Can one give some corresponding result for surfaces? □

LM 22 (1966). If all the faces of a convex polyhedron have central symmetry, at least 8 of the vertices are of order 3. If there are n vertices, at least how many must have order 3? □

LM 23 (1966). If all the faces of a convex polyhedron have central symmetry, at least 6 of the faces are parallelograms. If there are F faces, at least how many are parallelograms? □

This problem is equivalent to Sylvester's problem on the number of ordinary lines determined by n points in the plane. For a survey of Sylvester's problem and its generalizations see Borwein and Moser [12].

LM 24 (1966). Is there a nonpathological dissection of the plane (considered as a point set) into n congruent connected pieces? □

LM 25 (1966). Estimate the "size" of the largest measurable point set in a large square, which does not determine unit distance. □

Erdős comments: L. Szekely has a very recent paper in Ars Combinatoria on this subject.

> L. Moser was in H. Robbins' office one morning. Leo asked "Are you going to the mathematics seminar at 3:30 this afternoon?" Robbins replied "Oh sure — can't you see how busy I am getting my work done so that I will be able to go." "You sound as though you are joking — are you?" asked Moser. "Are you going?" asked Robbins. "Yes" replied Moser. "Then" said Robbins, "at 3:30 you will know whether I am joking, and at 4:30 you will know the reason why."

LM 26 (1966). Given a polygon such that the smallest angle determined by triples of its vertices is θ, we can prove the existence of a fixed $c > 0$ such that any such polygon can be "dissected into a square" in fewer then $c^{1/\theta}$ triangular pieces. Improve this estimate. □

LM 27 (1966). Find the minimum number of pieces into which one can dissect the unit cube to give a unit cube in any other orientation by translations only. □

LM 28 (1966). Estimate the number of regular "tetrahedra" of edge 1 which can be placed in a unit "cube" in n-space. □

LM 29 (1966). What is the minimal volume common to 5 unit cylinders having concurrent axes? □

LM 30 (1966). Let $f \in L(-\infty, \infty)$, $0 \leq f \leq 1$, $\int_{-\infty}^{\infty} f = 1$. If

$$M(t) = M(f, t) = \int_{-\infty}^{\infty} f(x) f(x+t) \mathrm{d}x,$$

$$m = m(f) = \inf_{|t| \leq 1} M(t),$$

$$\mu = \inf_{f} (1 - m),$$

find or estimate μ. □

Erdős comments: Murdeshwar and I can only prove $\mu \geq 0.5892$. This problem is essentially due to Czipszer and the above estimate improves a previous estimate $\mu \geq 0.528$ due to Swierczkowski.

For a similar overlap problem see W. Moser [45].

LM 31 (1966). Given a "chessboard" in n dimension, 3 to a side, how many squares can one enter without getting "three in a row"? We can show how to enter $o(3^n/n)$ but probably one cannot enter $\varepsilon 3^n$ for any $\varepsilon > 0$ if $n > n_0(\varepsilon)$. Such a result would be stronger than Roth's theorem on arithmetic progressions. □

This problem has apparently been settled by Furstenberg, Wan, Katznelson.

LM 32 (1966). How to dissect a sphere into regions by n great circles, no three concurrent, so as to minimize the sum of the squares of the areas of the regions. □

> When Leo Moser was playing in the chess tournament in Toronto in 1946, a heckler was bothering the players. "Chess is a complete waste of time," he said. "It has no relation to any other branch of knowledge." "How about mathematics?" Moser asked. "I have studied mathematics for many years," he replied, "and know that chess has no relation to any of the four branches of mathematics." "What branches do you mean?" Moser asked. "You know", he replied, "addition, subtraction, multiplication and division."

LM 33 (1966). For an $n \times n$ chessboard (for beginners let it be a checkerboard) how many squares chosen at random must one enter before it becomes likely that the entered squares form a connected configuration? □

Erdős comments: Füredi has a paper on this subject.

LM 34 (1966). Let $\varrho = \varrho(\theta)$ define a convex region around the origin. Determine the smallest c such that $\int_0^\pi \varrho(\theta)\varrho(\theta+\pi)\mathrm{d}\theta > c$ will imply the existence of a nontrivial lattice point inside the region. □

LM 35 (1966). Estimate the largest $f = f(n)$ such that every convex polyhedron of n vertices has an orthogonal projection onto the plane with $f(n)$ vertices on the "outside". □

LM 36 (1966). Given a linear point set on (0,1) of measure $\frac{1}{2}$ (say). At least how large a portion of (0,1) can be covered by the union of n translates of the set? □

LM 37 (1966). Let $P_1, P_2, \ldots, P_n$ be points in 3-space and let $f(P) = \prod_{i=1}^n \overline{PP_i}$. Give a "geometric" proof that f cannot have a local maximum. □

LM 38 (1966). Let $0 < r < \frac{1}{2}$ and $f(r,\theta)$ be the longest segment, one end at O, which does not come closer than r to any (nonorigin) lattice point. It is known that $\max_\theta f(r,\theta)$ is rather close to $1/r$. (Polya.) Can one give precise information about this function and its discontinuities? How about higher dimensional case? □

LM 39 (1966). Let v be a vector and consider the n points $v, 2v, \ldots, nv$ reduced modulo 1, i.e., on the torus. Associate with each of the n points those points of the torus which are closer to it than to any of the other $n-1$ points. Is there a bound (independent of v and n) for the number of distinct "Dirichlet" regions so obtained? □

LM 40 (1966). Prove that the number of n-ominoes does not exceed 4^n. □

LM 41 (1966). Let $f(n)$ be the maximum number of points which one can pick in the unit n-cube so that all mutual distances are ≥ 1. Clearly $f(n) = 2^n$ for $n = 1, 2, 3$. Meir has proved that $f(4) = 17$ and several have shown that $\log f(n) \sim \frac{1}{2} n \log n$. Evaluate $f(5)$ and sharpen the asymptotic relation. □

LM 42 (1966). Any 5 or more great circles, no 3 concurrent, determine one spherical polygon of ≥ 5 sides. Sharpen or extend this result. □

LM 43 (1966). Given a closed curve of bounded curvature surrounding the origin, show that if it is magnified radially, then as the magnification factor tends to infinity, the distance of the curve to the closest lattice point tends to zero. Obtain quantitative versions of this result. □

LM 44 (1966). n points are chosen at random in a unit square. What is the expected number of sides of the Dirichlet regions which are determined? □

LM 45 (1966). Given a graph G in the plane with v vertices and E edges, all of length 1. Sharpen the inequality $E<\frac{1}{4}v(1+\sqrt{8v-7})$. □

LM 46 (1966). Given n great circles on a sphere, no 3 concurrent. If $n\equiv 0 \pmod 4$ there is no trip which visits all the countries once and only once and crosses only at simple boundary points. As $n\to\infty$ does the fraction of countries reachable in one "simple" trip diminish? What is the situation for $n\not\equiv 0 \pmod 4$? □

LM 47 (1966). In an $n\times n$ square lattice at least how many points must we pick so that all n^2 point will be visible from one of the chosen set? □

LM 48 (1966). Find or estimate the smallest $m=m(n)$ such that there exists an m-omimo which contains as a proper subset every n-omino. □

LM 49 (1966). What is the largest $f(n)$ such that every convex polyhedron of n vertices has a simple path along edges passing through $f(n)$ of the vertices? Moon and I have shown $f(n)<cn^{\log 2/\log 3}$. □

For a survey of results on the maximal length of simple circuits in certain 3-connected planar graphs see Grünbaum and Walther [33].

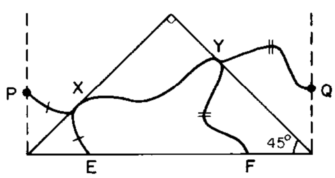
LM 50 (1966). Can every closed curve of length π be accomodated in a rectangle of area 4? □

Can every closed curve of length 2π be accomodated in a rectangle of area 4? □□

What is the largest number α such that every closed curve of length α can be covered by a translate (displacement; congruent copy) of an equilateral triangle of side 1? A triangle of sides a,b,c? Versions of this problem appear in Graham [32, p. 98] and Croft [18,19]. Solutions to these and related problems and similar unsolved problems appear in Besicovitch [9], Wetzel [59–61], Besicovitch and Rado [10], Kinney [38], Schaer [52], Schaer and Wetzel [53], Jones and Schaer [37], Poole and Gerriets [50,51], Chakerian and Klamkin [15].

Comments by John Wetzel:

"Gerriets and Poole [50,51] have what appears to be the smallest known convex cover for all arcs of length 1; it is a certain irregular pentagon with area less than 0.286.



$\circ\ K_n$ —

One can easily do better for the family of arcs of unit length that lie to one side of the line through their endpoints (and in particular for convex arcs). Indeed, encage such an arc in an isosceles right triangle of hypotenuse h with its ends on the hypotenuse and a point of contact on each leg. Reflecting the arcs XE and YF through the legs shows at once that $h \leq 1$, and it follows that the isosceles right triangle with hypotenuse 1 and area 0.25 is a cover for such arcs.

One can clip the top off (because tall curves can be turned), but the best I have been able to do reduces the area only to 0.2492. I suspect one can clip down to the height of the broadworm, which would leave an area of 0.2463, but I cannot argue it. The best known lower bound is still 0.2195 (the least area spanned by the broadworm and a straight segment), so the range remains rather wide."

Pal [48] showed that the equilateral triangle is the smallest plane convex set containing a unit segment in every direction; the analogous problem in higher dimensions remains unsolved. Related problems are considered by Eggleston [20], Besicovitch [9] and Wetzel [61]. The book by Falconer [25] also contains much related material.

Existence of solutions to the problem of the minimal universal cover for all arcs of length 1 and the minimal universal cover for all sets of diameter 1 (RPDG # 30) have been shown as corollaries to a more general existence theorem proved by Laidacker and Poole [40].

> Kummer, the German algebraist, was rather poor at arithmetic. Whenever he had occasion to do simple arithmetic in class he would get his students to help him. Once he had occasion to find 7×9. "7×9", he began, "7×9 is *er — ah — ah* — 7×9 is ···" 61, a student suggested. Kummer wrote 61 on the board. "Sir", said another student, "It should be 69." "Come, come, gentlemen," said Kummer, "it can't be both — it must be one or the other."

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A system of circles in the plane is said to form a k-fold covering if every point of the plane is contained in the interior of at least k circles. A 1-fold covering is simply called a covering.

(1) Does there exist a natural number k such that every k-fold covering of the plane with circles can be decomposed into 2 coverings?

Mani and I proved that this is true for coverings with equal circles, but it fails to hold in higher dimensions.

(2) Can one define a graph G whose vertex set is the points of the plane with integer coordinates, and which satisfies

$$|\mathrm{dist}_G(x,y)-\mathrm{dist}(x,y)|\leq K \quad \text{for every } x,y,$$

with K an absolute constant? Here $\mathrm{dist}_G(x,y)$ and $\mathrm{dist}(x,y)$ denote respectively the distance between x and y in the graph G (i.e., the length of shortest path connecting them in G) and the Euclidean distance between x and y.

(3) Let $\{S_i \mid i=1,2,\dots\}$ be a sequence of slabs in $\mathbb{R}^3$ with total thickness $\sum_{i=1}^{\infty} w(S_i)=+\infty$. Is it true that the unit ball can be covered by $\bigcup_{i=1}^{\infty} S_i'$ where S_i' is an appropriate translate of S_i? (A slab of thickness w is the part of the space between two parallel planes at distance w from each other.)

Endre Makai Jr and I proved the corresponding statement in the plane. Several applications can be found in our paper that appeared in Studia Sci. Math. Hungar. 18 (1983) 435–459.

(4) Consider n points in the plane in general position. What is the maximum number of pairs that can be joined by straight line segments without creating k mutually crossing segments?

For $k=2$ the answer is obviously $\leq 3n-6$.

For $k=3$ the question was asked independently by several people (most recently by Elmo Welzl). Even if the points are in convex position, I cannot give an upper bound better than $cn \log n$.

D. Rappaport (Queen's University)

Let S be a set of n discs in the plane. The convex hull of S is defined as the smallest convex region containing S. The boundary of the convex hull of S consists of no more than $4(n-1)$ arcs of circles and straight line segments. Given "reasonable" distributions for the centres and radii of the discs, what is the expected number of arcs on the boundary of the convex hull of S?

Everything convex interests me.
H. Minkowski

Diane Souvaine (Rutgers University)

The disjoint shadows problem was posed by Lee and Preparata in the late seventies: given n disjoint line segments in the plane, is there a direction from which the entirely of each line segment can be seen? Another way of posing the problem is the following: given n disjoint line segments in the plane, is there a line L such that the orthogonal projections of the n line segments onto L are pairwise disjoint?

The second way of phrasing the problem suggests a solution. Consider the n segments in pairs ($O(n^2)$ of them) and for each pair the directions under which their projections onto an orthogonal line would *not* be disjoint. For each pair, those "forbidden" directions would form a single continuous interval—or possibly two such intervals—if you were to split the set of directions considered at, say, the vertical. Thus, it would be possible to sort the $O(n^2)$ forbidden intervals and union them in $O(n^2 \log n)$ time. The space complexity is $O(n^2)$. If the union is the entire universe, then the answer to the disjoint shadows problem is "NO". If there is some interval which is not forbidden, then every direction in that interval is a "YES".

A different method of solving this problem involves dualization. If we take each of the original n segments and apply the dual transform $(x, y) \mapsto y = ax + b$ and $y = cx + d \mapsto (-c, d)$, then each segment is mapped to a double wedge. Regions where two double wedges intersect correspond to "forbidden" intervals, and viewing directions correspond to vertical lines. Thus, if there is some vertical line $x = m$ whose intersections with all of the double wedges are all pairwise disjoint, then all of the original line segments have disjoint projections onto a line having slope $1/m$ in the primal plane.

Solving this problem in the dual is also straightforward. Sweep a vertical line across the plane from $x = -\infty$ to $x = +\infty$. The vertical line always encounters $2n + 1$ regions. During the sweep, it is easy to keep track of the number of regions currently intersected by the sweep line which are interior to more than $\perp$ double wedge. If at some point during the sweep, this number reaches O, then the answer to DSP is YES; otherwise, NO. The time complexity of this algorithm is also $O(n^2 \log n)$, but it can be implemented using $O(n)$ space.

Open problem (posed in Edelsbrunner and Souvaine, 1988). Can this problem be solved in $o(n^2 \log n)$ time? In particular, can some variant of the topological sweep procedure developed by Edelsbrunner and Guibas help to produce a better time bound? Topological sweep can be used to solve the distinct shadows problem. Consider the arrangement in the dual plane created by the n double wedges. Sweeping this arrangement topologically can detect all forbidden x-intervals in $O(n^2)$ time. These intervals (($O(n^2)$ of them) then can be sorted and unioned in $O(n^2 \log n)$ time to determine whether any nonforbidden intervals remain. Alternately, as each forbidden interval is found, insert it into a balanced tree, immediately unioning any adjacent or overlapping intervals. Whenever a nonforbidden interval lies to the left of the min-x of the sweepline, stop and report "YES". If, having completed the sweep, the entire interval $(-\infty, \infty)$ is forbidden, report "NO". The tree contains at most $O(n^2)$ intervals and has at most $O(\log n)$ depth. Thus this algorithm has a worst case bound of $O(n^2 \log n)$ time and $O(n^2)$ space. But the order of the intervals as they are detected is not random. Is it possible to prove that the number of extant forbidden intervals at any time during the sweep can be kept small, and thus improve the time and space complexities cited above?

Subhash Suri (Bellcore)

Travelling salesman tour with minimum area. Given a set S of n points in the plane, we want to find a Hamiltonian circuit through S so that the area enclosed by the circuit is minimized. It is important to require that the circuit be simple (i.e., not self-intersecting). What is the complexity of this problem?

What bounds can be given for finding an approximation of the optimal solution?

Note: Recall that the standard travelling salesman problem in the plane minimizes the perimeter of the circuit, and, in that case, doubling of the minimum spanning tree gives an approximation of the optimal tour, within a factor of two.

G. Toussaint (McGill University)

Given a simple polygon P with n vertices, provide an algorithm for determining the points in P that form the *area-visibility-center* of P. A point x in P is in the visibility center if it maximizes the area of P visible from x. Two points x and y in P are *visible* if the line segment $[x, y]$ lies in P. Another version of the problem asks for the *perimeter-visibility-center* of P, i.e., the points that maximize the total length of the boundary of P that is visible.

J. Urrutia (University of Ottawa)

Illumination problems

The following problems arose from some recent work with J. Czyzowicz, B. Gaujal, E. Rivera-Campo, I. Rival and J. Zaks.

Let F be a family of disjoint convex sets on the plane. How many lamps (idealized as points on the plane emitting light) are needed to completely illuminate the boundaries of the elements of F?

It is known that for any family of n disjoint plane convex sets $4n-7$ lamps are always sufficient and occasionally necessary (Urrutia, Zaks).

(1) Is it true that any family of n disjoint line segments can be illuminated with at most $[n/2]$ lamps?

(2) Is it true that any family of n rectangles (with sides parallel to the coordinate axes) can be illuminated with $n+c$ lamps?

At the moment, we know that $[2n/3]$ lamps are always sufficient for 1 (Czyzowicz, Rivero-Campo, Urrutia and Zaks).

For (2), an upper bound of $[4n/3]$ is known (Czyzowicz, Rivera-Campo and Urrutia).

J. Zaks (University of Haifa)

(1) Is it true that for every two rational points x and y in Q^d (d-rational space), $5 \leq d \leq 7$, satisfying $\|x-y\| < 2$, there exists a rational point z in Q^d such that

$$\|x-z\| = \|y-z\| = 1?$$

The answer is "false" for $d \leq 4$, and "true" for $d \geq 8$.

(2) Due to K. Bezdek.) Is there a bound c_d of the size of a family of homothetic non-overlapping convex sets in E^d, in which every two members meet? Is $c_d = 2^d$?

Rongyao Zhao (McGill University)

A *tangent point* between two circles is a point at which the circles share a tangent line. What is the maximum number of tangent points among n circles of different radii in the plane (they are allowed to overlap)? Or find a nontrivial upper bound on the number of such tangent points.